A polynomial extension of a problem of Diophantus

Katalin Gyarmati*

Abstract

The following extension of a problem of Diophantus is studied: For finite sets $\mathcal{A}, \mathcal{B}$ of positive integers and a fixed polynomial p, how many pairs (a, b) $(a \in \mathcal{A}, b \in \mathcal{B})$ can be given so that the product ab is "near" to $p(x)$ for some positive integer x.

2000 AMS Mathematics subject classification number: 11D45.

Key words and phrases: bipartite graphs, polynomial, problem of Diophantus.

1 Introduction

The Greek mathematician Diophantus of Alexandria noted that the rational numbers $\frac{1}{16}$, $\frac{33}{16}$, $\frac{17}{4}$, and $\frac{105}{16}$ have the following property: the product of any two of them increased by 1 is a square of a rational number. Later Fermat

*Research partially supported by Hungarian Scientific Research Grant OTKA T043631 and T043623.

found a set of four positive integers with the above property: $\{1, 3, 8, 120\}$ (see [5]). A finite set $\mathcal{A}$ of integers is called a Diophantine n-tuple if $|\mathcal{A}| = n$ and $aa' + 1$ is a perfect square for all different elements a and a' of $\mathcal{A}$. The first absolute upper bound for the size of Diophantine tuples was given by A. Dujella [4], [6] and very recently he proved that there is no Diophantine 6-tuple, and there are only finitely many Diophantine 5-tuples [7]. Yann Bugeaud and A. Dujella [2] extended the problem for higher power.

In [3] the following related problem was studied: for an arbitrary set $\mathcal{A}$ at most how many pairs (a, a') exist with $a, a' \in \mathcal{A}$, $a \neq a'$, $aa' + 1 = x^n$. It is clear that if the number of these pairs is less than $|\mathcal{A}|^2 - |\mathcal{A}|$ than this implies that $\mathcal{A}$ is not a Diophantine $|\mathcal{A}|$-tuple. Indeed, there we were able to prove the following upper bound: *For $n \geq 2$, $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$, $|\mathcal{A}| \geq |\mathcal{B}|$ let*

$$S = |\{(a, b) : \ a \in \mathcal{A}, \ b \in \mathcal{B} \ ab + 1 = x^n, \ x \in \mathbb{N}\}|,$$

then

a) if $n = 2$ and $\mathcal{A} = \mathcal{B}$ then $S \leq \ 0.8\,|\mathcal{A}|^2$,

b) if $n = 3$ then $S \leq 16.27\,|\mathcal{A}|\,|\mathcal{B}|^{2/3}$

c) if $n \geq 4$ then $S \leq 11.93\,|\mathcal{A}|\,|\mathcal{B}|^{1/2}$

In particular, if $|\mathcal{A}| = |\mathcal{B}|$ then we obtain that the number of pairs (a, b) such that $ab + 1$ is a k-th power for a fixed $k \geq 4$ is $\leq 11.93\,|\mathcal{A}|^{3/2}$.

In the present paper we will study the case of general polynomials $p(x)$ in the place of x^n, and give an upper bound for the number of pairs (a, b) where ab is "around" $p(x)$ for a positive integer x. (Another polynomial extension of the problem was studied by A. Dujella and F. Luca [8]) This question was studied by H. Iwaniec and A. Sárközy [14] from the opposite side. They proved that for all positive constant c_1 there exists a constant c_2 (depending on c_1) such that if $\mathcal{A} \subseteq \{N, N+1, \dots, 2N\}$, $\mathcal{B} \subseteq \{N, N+1, \dots, 2N\}$ with $|\mathcal{A}| \geq c_1 N$, $|\mathcal{B}| \geq c_1 N$, then there exist integers a, b, x with $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $|ab - x^2| \leq c_2 (x \log x)^{1/2}$, so that ab is "near square". Wen-Guang Zhai [18] extended the problem to k sequences and k-th powers.

When $p(x) = cx^k$, there exist sets $\mathcal{A}$ and $\mathcal{B}$ (e.g. $A = \{y^k : \ y \in \mathbb{N}\}$, $B = \{cz^k : \ z \in \mathbb{N}\}$) such that ab is always of the form of $p(x)$, $x \in \mathbb{N}$. Therefore in this case it is not possible to give any non-trivial upper bound for the number of the pairs. However, under certain conditions on $p(x)$ we will be able to give an upper bound.

Theorem 1 *Let $p(x) = r_n x^n + r_m x^m + \cdots + r_0 \in \mathbb{R}[x]$ be a polynomial with $r_n > 0$, $0 \leq m \leq n-2$, $r_m \neq 0$. Suppose that $\alpha \in \mathbb{Q}$, $s \in \mathbb{Z}$ with $s \leq \min\{m-2, n/2\}$, $K \in \mathbb{R}$ with $K \leq r_n^{1/n}/6$ if $s = n/2$, $\mathcal{A}, \mathcal{B} \subseteq \{N, N+1, \dots, 2N\}$, and let*

$$S = |\{(a,b) : \ a \in \mathcal{A}, b \in \mathcal{B}, \ |ab - p(x)| \leq K x^s \textit{ for an } x \textit{ with } x + \alpha \in \mathbb{N}\}| .$$

Then

$$S \ll |\mathcal{A}| \, |\mathcal{B}|^{1/2} + |\mathcal{B}|$$

always holds.

Throughout the paper we use the notations $\ll, \gg, \asymp$ and O in the sense that the implied constant factor may depend only on the polynomial p and the real numbers K and α defined in Theorem 1 and 2. More exactly we say that $f \ll g$ or $f = O(g)$ if there exists a positive constant c_3 such that $|f| < c_3 |g|$, and we say that $f \asymp g$, if there exist two positive constants c_4

and c_5 such that $c_4 f \leq g \leq c_5 f$, where the constants c_3, c_4, c_5 depend only on the polynomial p and the real numbers K and α, but nothing else (these constants does not depend on the sets $\mathcal{A}$ and $\mathcal{B}$ or the integers N and M where $\mathcal{A} \subseteq \{N, N+1, \dots,\}$, $\mathcal{B} \subseteq \{M, M+1, \dots\}$). Moreover, throughout the paper $c_6, c_7, c_8 \dots$ will also denote positive constants whose value may depend only on the polynomial p and the real numbers K and α but nothing else.

The following two examples show that the conditions $s \leq n/2$ and $\mathcal{A}, \mathcal{B} \subseteq [N, N+1, \dots, 2N]$ are necessary. First consider the case $s > n/2$. Let $y, x \in \mathbb{N}$, $y^2 = p(x)$ and $\mathcal{A} = \mathcal{B} = \{a : \ |a - y| \leq cy^{\epsilon}, \ a \in \mathbb{N}\}$. Then for $a \in \mathcal{A}$, $b \in \mathcal{B}$ we have

$$|ab - p(x)| = \left|ab - y^2\right| \leq c^2 y^{2\epsilon} + 2cy^{1+\epsilon} \ll x^s$$

if ϵ is small enough and $s > n/2$. Thus

$$S = |\mathcal{A}| \, |\mathcal{B}| \, .$$

So, indeed, the condition $s \leq n/2$ is necessary.

On the other hand assume that $s \leq n/2$, v and $0 \leq \ell \leq v$ are fixed integers, $p(x) \in \mathbb{Q}[x]$, c_6, c_7 are positive constants, $\mathcal{A} = \{p(x)/v : \ x \in \mathbb{N}, \ c_6 < x < c_7 \, (v/\ell)^{1/(n-s)}, \ v \mid p(x)\}$, $\mathcal{B} = \{v, v+1, \dots, v+\ell\}$. If the constant c_6 is large enough then $p(x) \asymp x^n$, and in this case for small enough constant c_7 we get $S = |\mathcal{A}| \, |\mathcal{B}|$ and $\mathcal{A} \subseteq [1, 2, \dots v]$, $\mathcal{B} \subseteq [v, v+1, \dots, 2v]$. Thus in Theorem 1 the condition $\mathcal{A}, \mathcal{B} \subseteq [N, N+1, \dots, 2N]$ is also important. In this example, it is very difficult question to give estimates for the size of $\mathcal{A}$, however the conjecture is that there exists a polynomial p for which the cardinality of $\mathcal{A}$ is large. Considering the possible values of the polynomial $p(x)$ which are around ab in our example: $x \in \mathbb{N}$ and there exist $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $|ab - p(x)| < Kx^s$, we see that the number of these x's is $|\mathcal{A}|$. Thus the values of the polynomial $p(x)$ with integer x, around a product ab, is much less than the number of all pairs (a, b) with $a \in \mathcal{A}$, $b \in \mathcal{B}$ which is $|\mathcal{A}|\,|\mathcal{B}|$. Generally, we will prove the following:

Theorem 2 *Let* $p(x) = r_n x^n + r_m x^m + \cdots + r_0 \in \mathbb{R}[x]$ *be a polynomial with* $r_n > 0$, $0 \le m \le n-2$, $r_m \ne 0$. *Suppose that* $\alpha \in \mathbb{Q}$, $s \in \mathbb{Z}$ *with* $s \le \min\{m-2, n/2\}$, $K \in \mathbb{R}$ *with* $K \le r_n^{1/n}/6$ *if* $s = n/2$, $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$, *and let*

$$S' = |x:\ a \in \mathcal{A}, b \in \mathcal{B},\ |ab - p(x)| \le Kx^s \text{ for an } x \text{ with } x + \alpha \in \mathbb{N}\}|\,.$$

If one of the following 3 conditions holds

a) $\mathcal{A} \subseteq \{N, N+1, \ldots, 2N\}$, $\mathcal{B} \subseteq \{M, M+1, \ldots, 2M\}$,

b) $m \le n-3$ *and* $\mathcal{A} \subseteq \{N, N+1, \ldots, N^2\}$, $\mathcal{B} \subseteq \{M, M+1, \ldots, M^2\}$,

c) $n \ge 4$, $m < n/\alpha_n$, *where* $\alpha_n = \max\{\frac{3n-2}{2(n-3)}, \frac{2(n-1)}{n-2}\}$, $|\mathcal{A}| \ge |\mathcal{B}|$,

then $S' \ll |\mathcal{A}|\,|\mathcal{B}|^{1/2} + |\mathcal{B}|\,.$

Unfortunately we have not been able to prove an upper bound for every polynomial without restrictions of the size of the sets $\mathcal{A}$ and $\mathcal{B}$. However, one of $a)$ and $b)$ holds for every polynomial, since all polynomials can be written in the form $r_n(x+\alpha)^n + r_{n-2}(x+\alpha)^{n-2} + r_{n-3}(x+\alpha)^{n-3} + \cdots + r_0$.

In Corollary 1 we study the number of products ab which are of the form of $p(x)$ exactly, and this result will follow from the proofs of Theorems 1 and 2.

Corollary 1 *Suppose that the conditions of Theorem 2 hold and let*

$$S = |\{(a,b): \ a \in \mathcal{A}, b \in \mathcal{B}, \ ab = p(x) \text{ for an } x \text{ with } x + \alpha \in \mathbb{N}\}| \, .$$

Then we have

$$S \ll |\mathcal{A}| \, |\mathcal{B}|^{1/2} + |\mathcal{B}| \, .$$

Throughout this paper, for a graph G, $v(G)$ denotes the number of the vertices, $e(G)$ the number of the edges of G. C_k denotes the cycle of length k, $K_{r,t}$ is the complete bipartite graph with r and t vertices in its classes.

2 Lemmas

Theorems 1, 2 and Corollary 1 are based on graph theory. In the next section we will define graphs whose vertices are the elements of $\mathcal{A}$, $\mathcal{B}$, and two vertices $a \in \mathcal{A}$, $b \in \mathcal{B}$ are joined if ab is around $p(x)$ for an $x + \alpha \in \mathbb{N}$. We will give an upper bound for the number of the edges using the fact that these graphs do not contain "large" bipartite complete subgraph.

In this section, we study the case when the graph contains a cycle of length 4. More exactly, we suppose that there exist 4 integers a, b, c, d as described in Hypothesis 1.

Hypothesis 1 *Let* $p(x) = r_n x^n + r_m x^m + r_{m-1} x^{m-1} + \cdots + r_0 \in \mathbb{R}[x]$ *be a polynomial with* $r_n > 0$, $0 \leq m \leq n-2$, $r_m \neq 0$ *and let* $\alpha \in \mathbb{Q}$, $s \in \mathbb{Z}$ *with* $s \leq \min\{m-2, n/2\}$, $K \in \mathbb{R}$ *with* $K \leq r_n^{1/n}/6$ *if* $s = n/2$. *Suppose that there* *exist* 4 *integers* $a < b$, $c < d$ *with*

$$|ac - p(x)| \le Kx^s, \ |ad - p(v)| \le Kv^s,$$

$$|bc - p(z)| \le Kz^s, \ |bd - p(y)| \le Ky^s,$$

for some $x + \alpha, y + \alpha, z + \alpha, v + \alpha \in \mathbb{N}$.

Throughout this section we use the notation of Hypothesis 1. Using that $x+\alpha, y+\alpha, z+\alpha, v+\alpha$ are integers we will prove the following two lemmas which are the main result of the section.

Lemma 1 *There exist constants* $c_8 > 1, c_9$ *such that if* a, b, c, d *are large enough (depending on* $p(x)$, K, α*) and* $xy - vz \neq 0$ *then we have*
a) $c_8 ac < bd$ *if* $m = n - 2$,
b) $c_9 (ac)^{n-m-1} < bd$ *if* $m \le n - 3$.

Lemma 2 *There exists a constant* c_{10} *such that if* a, b, c, d *are large enough,* $xy - vz = 0$ *and one of the following 3 condition holds*
a) $s < 0$,
b) $s < n/2$, $2b > c$ *and* $2d > a$,
c) $s = n/2$, $K \le r_n^{1/n}/6$, $2b > c$ *and* $2d > a$,
then we have

$$c_{10} (ac)^{m-s} < bd.$$

First we need estimates for the exact value of x, y, z, v.

Lemma 3 *Let* $p(x) = r_n x^n + r_m x^m + r_{m-1} x^{m-1} + \cdots + r_0 \in \mathbb{R}[x]$ *be a polynomial with* $r_n > 0$, $0 \le m \le n - 2$, $r_m \neq 0$ *and let* $K \in \mathbb{R}$, $s \in \mathbb{Z}$ *with*

$s \le m$. Then there exist a constant c_{11} and $a_{-1}, a_{n-m-1}, a_{n-m}, \dots, a_{n-s-2}$ such that if $y > c_{11}$, $x > 0$ and

$$|y - p(x)| < Kx^s \tag{1}$$

then we have

$$x = a_{-1}y^{1/n} + \frac{a_{n-m-1}}{y^{(n-m-1)/n}} + \frac{a_{n-m}}{y^{(n-m)/n}} + \dots + \frac{a_{n-s-2}}{y^{(n-s-2)/n}} + O(\frac{1}{y^{(n-s-1)/n}}) \tag{2}$$

with $a_{-1} = \frac{1}{r_n^{1/n}}$.

Proof of Lemma 3. If y is large enough then from (1) we get

$$x = O(y^{1/n}). \tag{3}$$

Then

$$\left| x - \frac{y^{1/n}}{r_n^{1/n}} \right| = \frac{\left| x^n - \frac{y}{r_n} \right|}{\left| x^{n-1} + x^{n-2}\frac{y^{1/n}}{r_n^{1/n}} + \dots + \frac{y^{(n-1)/n}}{r_n^{(n-1)/n}} \right|} \le \frac{|r_n x^n - y|}{r_n x^{n-1}}. \tag{4}$$

By the triangle-inequality we have

$$|r_n x^n - y| \le |r_n x^n - p(x)| + |p(x) - y| = O(x^m) + O(x^s) = O(x^m).$$

From this, (3) and (4) we obtain

$$\left| x - \frac{y^{1/n}}{r_n^{1/n}} \right| = O(\frac{1}{x^{n-m-1}}) = O(\frac{1}{y^{(n-m-1)/n}}). \tag{5}$$

From the Taylor-formula, by induction on the number of the constants (which is $m-s+1$), it is easy to prove that there exist constants $a_{-1}, a_{n-m-1}, \dots, a_{n-s-2}$ such that if

$$x_0 = a_{-1}y^{1/n} + \frac{a_{n-m-1}}{y^{(n-m-1)/n}} + \frac{a_{n-m}}{y^{(n-m)/n}} + \dots + \frac{a_{n-s-2}}{y^{(n-s-2)/n}} \tag{6}$$

then

$$p(x_0) = y + O(y^{s/n}).$$

On the other hand if (1) holds then by (3) we have

$$p(x) = y + O(x^s) = y + O(y^{s/n}).$$

Using the Lagrange theorem we get

$$O(y^{s/n}) = |p(x) - p(x_0)| = p'(\xi)\,|x - x_0| \tag{7}$$

for some $\xi \in [x, x_0]$. By (5) and (6) we have

$$\xi = \frac{y^{1/n}}{r_n^{1/n}} + O(\frac{1}{y^{(n-m-1)/n}}).$$

Thus $p'(\xi) \gg y^{(n-1)/n}$. From (7) we obtain:

$$O(y^{s/n}) \gg y^{(n-1)/n}\,|x - x_0|\,.$$

Then:

$$|x - x_0| = O(\frac{1}{y^{(n-s-1)/n}})$$

which completes the proof of Lemma 3.

By Lemma 3 we have that there exist constants $a_{-1}, a_{n-m-1}, \ldots, a_{n-s-2}$

(which may depend on the coefficients of $p(x)$) such that

$$
\begin{aligned}
x &= a_{-1}(ac)^{1/n} + \frac{a_{n-m-1}}{(ac)^{(n-m-1)/n}} + \frac{a_{n-m}}{(ac)^{(n-m)/n}} + \cdots + \frac{a_{n-s-2}}{(ac)^{(n-s-2)/n}} \\
&\quad + O(\frac{1}{(ac)^{(n-s-1)/n}}), \\
v &= a_{-1}(ad)^{1/n} + \frac{a_{n-m-1}}{(ad)^{(n-m-1)/n}} + \frac{a_{n-m}}{(ad)^{(n-m)/n}} + \cdots + \frac{a_{n-s-2}}{(ad)^{(n-s-2)/n}} \\
&\quad + O(\frac{1}{(ad)^{(n-s-1)/n}}), \\
z &= a_{-1}(bc)^{1/n} + \frac{a_{n-m-1}}{(bc)^{(n-m-1)/n}} + \frac{a_{n-m}}{(bc)^{(n-m)/n}} + \cdots + \frac{a_{n-s-2}}{(bc)^{(n-s-2)/n}} \\
&\quad + O(\frac{1}{(bc)^{(n-s-1)/n}}), \\
y &= a_{-1}(bd)^{1/n} + \frac{a_{n-m-1}}{(bd)^{(n-m-1)/n}} + \frac{a_{n-m}}{(bd)^{(n-m)/n}} + \cdots + \frac{a_{n-s-2}}{(bd)^{(n-s-2)/n}} \\
&\quad + O(\frac{1}{(bd)^{(n-s-1)/n}}). \qquad (8)
\end{aligned}
$$

Thus

$$
\begin{aligned}
xy - vz = &\sum_{\substack{i<j\in\{-1/n,(n-m-1)/n,\\ (n-m)/n,\dots,(n-s-2)/n\}}} a_i a_j \left(b^{-i}a^{-j} - b^{-j}a^{-i}\right)\left(d^{-i}c^{-j} - d^{-j}c^{-i}\right) \\
&+ O(\frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}}).
\end{aligned}
$$

Then we have

Lemma 4

$$
\begin{aligned}
|xy - vz| \ll &\left(\frac{b^{1/n}}{a^{(n-m-1)/n}} - \frac{a^{1/n}}{b^{(n-m-1)/n}}\right)\left(\frac{d^{1/n}}{c^{(n-m-1)/n}} - \frac{c^{1/n}}{d^{(n-m-1)/n}}\right) \\
&+ \frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}}, \qquad (9)
\end{aligned}
$$

and there exists a constant c_{12} *such that*

$$|xy - vz| \gg \left(\frac{b^{1/n}}{a^{(n-m-1)/n}} - \frac{a^{1/n}}{b^{(n-m-1)/n}}\right)\left(\frac{d^{1/n}}{c^{(n-m-1)/n}} - \frac{c^{1/n}}{d^{(n-m-1)/n}}\right) - c_{12}\frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}}. \tag{10}$$

Before proving Lemma 4, we remark that the main tool in the proofs of Lemma 1 and Lemma 2 is the fact that $xy - zv$ is a rational number. More precisely, let $\alpha = r/q$ where $r, q \in \mathbb{Z}$, $(r, q) = 1$. $x, y, z, v \in \mathbb{N} + \alpha$ therefore $q^2 |xy - vz|$ is an integer, however considering (9) we would think that if a, b, c, d are large enough (depending on $p(x)$, K, α), then $q^2 |xy - vz|$ is usually smaller than 1.

Proof of Lemma 4. Using that $a < b$, by studying the derivatives we obtain that the function $x \mapsto b^{-x}a^{-j} - b^{-j}a^{-x}$ in $[-\infty, j - 1/n]$ and the function $x \mapsto b^{-i}a^{-x} - b^{-x}a^{-i}$ in $[i + 1/n, \infty]$ are decreasing. Thus the largest number of the set $\{b^{-i}a^{-j} - b^{-j}a^{-i} :\ i < j \in \{-1/n, (n-m-1)/n, \ldots, (n-s-2)/n\}\}$ is $\frac{b^{1/n}}{a^{(n-m-1)/n}} - \frac{a^{1/n}}{b^{(n-m-1)/n}}$ and the second largest is $\frac{b^{1/n}}{a^{(n-m)/n}} - \frac{a^{1/n}}{b^{(n-m)/n}}$. So we have

$$\begin{aligned}|xy - vz| = a_{-1}a_{n-m-1}&\left(\frac{b^{1/n}}{a^{(n-m-1)/n}} - \frac{a^{1/n}}{b^{(n-m-1)/n}}\right)\left(\frac{d^{1/n}}{c^{(n-m-1)/n}} - \frac{c^{1/n}}{d^{(n-m-1)/n}}\right)\\ &+ O(\left(\frac{b^{1/n}}{a^{(n-m)/n}} - \frac{a^{1/n}}{b^{(n-m)/n}}\right)\left(\frac{d^{1/n}}{c^{(n-m)/n}} - \frac{c^{1/n}}{d^{(n-m)/n}}\right))\\ &+ O(\frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}})\end{aligned} \tag{11}$$

Next we claim that

$$\left(\frac{b^{1/n}}{a^{(n-m-1)/n}} - \frac{a^{1/n}}{b^{(n-m-1)/n}}\right) \geq \frac{a^{1/n}}{2}\left(\frac{b^{1/n}}{a^{(n-m)/n}} - \frac{a^{1/n}}{b^{(n-m)/n}}\right) \tag{12}$$

whence, since this statement is also true for $c < d$ in place of $a < b$, Lemma 4 follows trivially from (11). Let us see the proof of (12). Indeed, by $a < b$ we have

$$\frac{a^{1/n}}{2(b^{1/n} - a^{1/n}/2)} > \left(\frac{a}{b}\right)^{2/n} \geq \left(\frac{a}{b}\right)^{(n-m+1)/n},$$

which is equivalent with (12). Thus we have proved Lemma 4.

Proof of Lemma 1

By Lemma 4 we have

$$\begin{aligned} 1 \leq q^2\,|xy - vz| \ll & \left(\frac{b^{1/n}}{a^{(n-m-1)/n}} - \frac{a^{1/n}}{b^{(n-m-1)/n}}\right)\left(\frac{d^{1/n}}{c^{(n-m-1)/n}} - \frac{c^{1/n}}{d^{(n-m-1)/n}}\right) \\ & + \frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}} = \frac{(bd)^{1/n}}{(ac)^{(n-m-1)/n}} + \frac{(ac)^{1/n}}{(bd)^{(n-m-1)/n}} \\ & - \frac{(ad)^{1/n}}{(bc)^{(n-m-1)/n}} - \frac{(bc)^{1/n}}{(ad)^{(n-m-1)/n}} + \frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}}. \end{aligned}$$

Since

$$\begin{aligned} \frac{(ac)^{1/n}}{(bd)^{(n-m-1)/n}} &\leq \min\{\frac{(ad)^{1/n}}{(bc)^{(n-m-1)/n}}, \frac{(bc)^{1/n}}{(ad)^{(n-m-1)/n}}\}, \\ \frac{1}{(abcd)^{(n-m-2)/2n}} &\leq \max\{\frac{(ad)^{1/n}}{(bc)^{(n-m-1)/n}}, \frac{(bc)^{1/n}}{(ad)^{(n-m-1)/n}}\}, \end{aligned}$$

we have

$$1 \leq c_{13}\left(\frac{(bd)^{1/n}}{(ac)^{(n-m-1)/n}} - \frac{1}{(abcd)^{(n-m-2)/2n}} + \frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}}\right).$$

for some constant c_{13}. Thus if ac is large enough then

$$\begin{aligned} (ac)^{(n-m-1)/n} + c_{13}\frac{(ac)^{(n-m)/n}}{(bd)^{(n-m-2)/2n}} &\leq c_{13}(bd)^{1/n} + O(\frac{(bd)^{1/n}}{(ac)^{(m-s)/n}}) \\ &\leq (c_{13} + 1/2)\,(bd)^{1/n}. \end{aligned}$$

Thus if $m = n - 2$ we get

$$\left(\frac{c_{13}+1}{c_{13}+1/2}\right)^n ac \le bd,$$

and when $m \le n-3$ we get

$$(ac)^{n-m-1} < (c_{13}+1/2)^n\, bd,$$

which completes the proof of Lemma 1.

Proof of Lemma 2

In order to prove Lemma 2 we need the following lemma.

Lemma 5 *There exists a constant c_{14} such that if a, b, c, d are large enough, $c_{14}(ac)^{m-s} \ge bd$ and $xy - vz = 0$, then $\{x, y\} = \{z, v\}$.*

Proof of Lemma 5 By Lemma 4 there exists a constant c_{12} such that for $a < b$, $c < d$ we have

$$\begin{aligned} 0 = |xy - vz| &\gg \left(\frac{b^{1/n}}{a^{(n-m-1)/n}} - \frac{a^{1/n}}{b^{(n-m-1)/n}}\right)\left(\frac{d^{1/n}}{c^{(n-m-1)/n}} - \frac{c^{1/n}}{d^{(n-m-1)/n}}\right) \\ &\quad - c_{12}\frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}} \\ &\gg \frac{1}{(ac)^{(n-m-1)/n}}(b^{1/n} - a^{1/n})(d^{1/n} - c^{1/n}) - c_{12}\frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}}. \end{aligned}$$

Since

$$|x + y - v - z| = (b^{1/n} - a^{1/n})(d^{1/n} - c^{1/n}) + O(\frac{1}{(ac)^{(n-m-1)/n}})$$

we obtain that there exists a constant c_{15} such that

$$\begin{aligned} 0 = |xy - vz| &\gg \frac{1}{(ac)^{(n-m-1)/n}}\left(|x + y - v - z| - c_{15}\frac{1}{(ac)^{(n-m-1)/n}}\right) \\ &\quad - c_{12}\frac{(bd)^{1/n}}{(ac)^{(n-s-1)/n}}. \end{aligned}$$

Whence by $m \le n-2$, $s \le m-2$ and $bd \le c_{14}(ac)^{m-s}$ with suitable constant c_{14} we obtain

$$q\,|x+y-v-z| < O(\frac{(bd)^{1/n}}{(ac)^{(m-s)/n}}) < 1.$$

$q\,|x+y-v-z|$ is an integer so if a and c are large enough, we have

$$x+y-v-z=0.$$

$xy = zv$, $x+y = z+v$ thus x, y and v, z are the roots of the same polynomial of degree 2 which completes the proof of Lemma 5.

Now we return to the proof of Lemma 2. Suppose that contrary to the statement $c_{14}(ac)^{m-s} \ge bd$. Then by Lemma 5 we have $\{x, y\} = \{z, v\}$. By symmetry reasons we may suppose that $x = v$ and $y = z$. By (8) we have $y = z = \frac{1}{r_n^{1/n}}(bc)^{1/n} + O((bc)^{(n-m-1)/n})$, so

$$|bc - p(y)| < Ky^s < \frac{2K}{r_n^{1/n}}(bc)^{s/n},$$
$$|bd - p(y)| < Ky^s < \frac{2K}{r_n^{1/n}}(bc)^{s/n},$$

From the triangle-inequality

$$|b| \le |bd - bc| \le \frac{4K}{r_n^{1/n}}(bc)^{s/n} \tag{13}$$

If $s < 0$ and b, c are large enough then $|b| \le 1/2$ which proves part a). In order to prove parts b) and c) from (13) we obtain:

$$1 \le |d-c| \le \frac{4K}{r_n^{1/n}}\left(\frac{c}{b}\right)^{(n-s)/n} c^{-(n-2s)/n}. \tag{14}$$

If $s < n/2$ and c is large enough we have

$$1 \le \frac{1}{2}\left(\frac{c}{b}\right)^{(n-s)/n}$$

which contradicts the condition $2b > c$ thus we have also proved part b). Finally if $s = n/2$ and $K \leq r_n^{1/n}/6$ then we have

$$1 \leq \frac{4K}{r_n^{1/n}} \left(\frac{c}{b}\right)^{1/2} \leq \frac{2}{3} \left(\frac{c}{b}\right)^{1/2}$$

which completes the proof of Lemma 2.

3 Completions of the proofs of the theorems.

Proof of Theorem 1. We cover the interval $[N, N+1, \ldots, 2N]$ by disjoint subintervals of the form $[z, c_{16}^{1/2} z]$. Then we have less than $\lceil 2 \log 2 / \log c_{16} \rceil$ subintervals. Define the bipartite graph G in the following way:

Consider two arbitrary subintervals $I_1 = [z_1, c_{16}^{1/2} z_1]$ and $I_2 = [z_2, c_{16}^{1/2} z_2]$. Graph G has two classes, one contains as vertices the elements of $\mathcal{A} \cap I_1$, while the others contains the elements of $\mathcal{B} \cap I_2$. There is an edge between the vertices $a \in \mathcal{A} \cap I_1$ and $c \in \mathcal{B} \cap I_2$ if and only if there exist $x + \alpha \in \mathbb{N}$ such that

$$|ac - p(x)| \leq K x^s.$$

By Lemma 1 and Lemma 2 b) and c) if this graph contains a C_4, whose vertices are a, b, c, d, then in both cases $xy - zv = 0$ or $xy - zv \neq 0$, we have $c_{16} ac < bd$ for $c_{16} \stackrel{\text{def}}{=} \min\{c_8, c_9, c_{10}\}$. But this contradicts $a, b, \in I_1$, $c, d \in I_2$ since $c_{16} ac \geq c_{16} z^2 \geq bd$. Thus this graph does not contain C_4. We will need the following lemma.

Lemma 6 *There exists a constant c_{17} such that if $G(X, Y)$ is a bipartite graph with $|X| = m$, $|Y| = n$ without C_4, then for the number of the edges* *we have*

$$e(G) \le c_{17} m n^{1/2} + n.$$

Proof of Lemma 6. See in [9] and [10].

Thus by Lemma 6, G has less edges than $|\mathcal{A}|\,|\mathcal{B}|^{1/2} + |\mathcal{B}|$ which proves the assertation for

$$|\{(a,b):\ a \in \mathcal{A} \cap I_1,\ b \in \mathcal{B} \cap I_2,\ |ab - p(x)| \le Kx^s \textit{ for an } x + \alpha \in \mathbb{N}\}|$$

We have at least $\lceil 2 \log 2 / \log c_{16} \rceil$ subintervals I_j and this completes the proof.

Proof of Theorem 2. If condition $a)$ holds in Theorem 2 we cover the intervals $[N, N+1, \dots, 2N]$, $[M, M+1, \dots, 2M]$ by disjoint subintervals of the form $[z, c_{17}^{1/2} z]$. When condition $b)$ holds we cover the intervals $[N, N+1, \dots, N^2]$, $[M, M+1, \dots, M^2]$ by disjoint subintervals of the form $[z, z^{3/2}]$. In case $c)$ we have only two subintervals $I_1 = I_2 = \mathbb{N}$. For given subintervals I_1, I_2 we define the graph G as in the proof of Theorem 1. We construct a graph G_0 from G by removing certain edges of G so that for each x such that there are $a \in \mathcal{A}$, $b \in \mathcal{B}$ with

$$|ab - p(x)| \le Kx^s. \tag{15}$$

there will be exactly one edge (a, b) left with this property. Suppose again G_0 contains a C_4, whose vertices are a, b, c, d. By Lemma 5 if $xy - zv = 0$, then, since x, y, z, v are different numbers, thus we have $c_{14}(ac)^{m-s} < bd$. Using this and Lemma 1 in case $a)$ and $b)$ we obtain that G_0 does not contain C_4 so as in the proof of Theorem 1 we get the statement.

Now consider the case c). We will prove that there exists a constant r such that graph G_0 does not contain a $K_{2,r}$ whose classes are $\{a,b\}$ and $\{d_1, d_2 \ldots, d_r\}$ where $a < b < d_1 < \cdots < d_r$.

$$ad_i = p(x) + O(x^s),\ bd_i = p(v) + O(v^s).$$

Then by the triangle inequality we have

$$|bp(x) - ap(y)| = O(\max\{bx^s, ay^s\}),$$
$$|bx^n - ay^n| = O(\max\{bx^m, ay^m\}).$$

Lemma 7 *If a, b and n are positive integers with $n \geq 3$ and c is a positive real number, then the inequality*

$$|ax^n - by^n| \leq c$$

has at most one positive integral solution (x, y) with

$$\max\{|ax^n|, |by^n|\} > \beta_n c^{\alpha_n},$$

where α_n and β_n are effectively computable positive constants satisfying

$$\alpha_3 = 9,\ \alpha_n = \max\left\{\frac{3n-2}{2(n-3)}, \frac{2(n-1)}{n-2}\right\} \text{ for } n \geq 4$$

and

$$\beta_3 = 1152.2,\ \beta_4 = 98.53,\ \beta_n < n^2 \text{ for } n \geq 5.$$

Proof of Lemma 7. This is Theorem 2.1 of [12].

By Lemma 7 there are only one x and y such that

$$\max\{|bx^n|, |av^n|\} > \beta_n \left(\max\{|bx^m|, |av^m|\}\right)^{\alpha_n}.$$

When this does not hold:

$$x^n \le \max\{x^n, v^n\} < \beta_n b^{\alpha_n} x^{\alpha_n m} \text{ or } v^n \le \max\{x^n, v^n\} < \beta_n a^{\alpha_n} v^{\alpha_n m},$$

whence we get for all d_i with one exception:

$$d_i < \max\{ad_i, bd_i\} \ll \max\{x^n, v^n\} < (\beta_n b^{\alpha_n})^{n/(n-\alpha_n m)}.$$

Let $ad_{i+1} = p(z) + O(z^s)$, $bd_{i+1} = p(y) + O(y^s)$. Then a, b, d_i, d_{i+1} form a C_4 in G_0, and since x, y, z, v are different, by Lemma 5 if $xy - zv = 0$ we have $c_{14}(ad_i)^{m-s} < bd_{i+1}$. Using Lemma 1 we get $c_9(ad_i)^{n-m-1} < bd_{i+1}$. Thus

$$c_{18}(ad_i)^{\min\{m-s,n-m-1\}} < bd_{i+1}$$

always holds so if $b < d_i$ and a are large enough we have

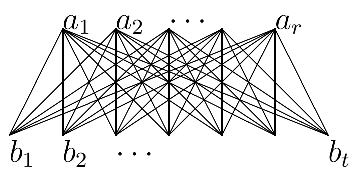
$$d_i^{\min\{m-s,n-m-1\}} < d_{i+1}.$$

Thus

$$d_1^{(r-1)(\min\{m-s,n-m-1\})} < d_{r-1} < (\beta_n b^{\alpha_n})^{n/(n-\alpha_n m)} < (\beta_n d_1)^{n/(n-\alpha_n m)}$$

which proves that the graph G_0 does not contain $K_{2,r}$ in the case c). Using the following lemma we will get the statement of Theorem 2.

Lemma 8 *Assume that $G(X,Y)$ is a bipartite graph with $|X| = m \le |Y| = n$, and the vertices are labeled by positive real numbers. Suppose that $G(X,Y)$ does not contain a $K_{r,t}$ subgraph G_0 for which*

$G_0 =$ 

with $a_i < b_j$ for all $1 \le i \le r$, $1 \le j \le t$ (where the a's belong to V_1 and the b's belong to V_2 or vice versa). Then G has at most

$$e(G) \le 2(t-1)^{1/r} n m^{1-1/r} + 2(r-1)n$$

edges.

Proof of Lemma 8. See in [3].

Proof of Corollary 1. Consider the same graph G again (but we do not remove the edges). Using Lemma 1 we can complete the proof of the corollary in the same way as in Theorem 2.

References

[1] B. Bollobás. *Extremal Graph Theory.* Academic Press, London-New York 1978.

[2] Y. Bugeaud, A. Dujella, *On a problem of Diophantus for higher powers*, Math. Proc. Cambridge Philos. Soc., 135 (2003), 1-10.

[3] Y. Bugeaud, K. Gyarmati, *On generalizations of a problem of Diophantus*, submitted.

[4] A. Dujella, *An absolute bound for the size of Diophantine m-tuples*, J. Number Theory 89 (2001), 126-150.

[5] A. Dujella, *On Diophantine quintuples*, Acta Arith. 81 (1997), 69-79.

[6] A. Dujella, *On the size of Diophantine m-tuples*, Math. Proc. Cambridge Philos. Soc. 132 (2002) 23-33.

[7] A. Dujella, *There are only finitely many Diophantine quintuples*, J. reine angew. Math., to appear.

[8] A. Dujella and F. Luca, *On a problem of Diophantus with polynomials*, Rocky Mountain J. Math., to appear.

[9] P. Erdős, *On sequences of integers no one of which divides the product of two others and on some related problems*, Tomsk. Gos. Univ. Ucen Zap., 2 (1938), 74-82.

[10] P. Erdős, *On some applications of graph theory to number theoretic problems*, Publ. Ramanujan Inst., 1 (1969), 131-136.

[11] P. Erdős, *Quelques problèmes de la théorie des nombres*, in: Monograph. Enseign. Math. 6, Geneva, 1963, 81-135.

[12] J.H. Evertse, *Upper bounds for the number of solutions of diophantine equations*, MCT 168, Mathematisch Centrum, Amsterdam, 1983.

[13] K. Gyarmati, *On a problem of Diophantus*, Acta Arith. 97.1 (2001), 53-65.

[14] H. Iwaniec, A. Sárközy, *On a multiplicative hybrid problem*, J. Number Theory 26, No. 1 (1987), 89-95.

[15] T. Kővári, V.T. Sós, P. Turán *On a problem of K. Zarankiewicz*, Colloquium Math. 3 (1954), 50-57.

[16] J. Rivat, A. Sárközy, C.L. Stewart, *Congruence properties of the* Ω*-function on sumsets*, Illinois J. Math. 43 (1999), no. 1, 1-18.

[17] W. Sierpiński, *A Selection of Problems in the Theory of Numbers*, Pergamon Press, 1964.

[18] Wen-Guang Zhai, *On a multiplicative hybrid problem*, Acta Arith. 71 (1995), 47-53.