Random polynomials in Legendre symbol sequences

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Abstract

It is important in cryptographic applications that the "key" used should be generated from a random seed. Thus, if the Legendre symbol sequence generated by a polynomial (as proposed by Hoffstein and Lieman) is used, that is

$$\left\{\left(\frac{f(1)}{p}\right),\left(\frac{f(2)}{p}\right),\left(\frac{f(3)}{p}\right),\dots,\left(\frac{f(p)}{p}\right)\right\},$$

then it is important to choose the polynomial f "almost" at random. Goubin, Mauduit, and Sárközy presented some not very restrictive conditions on the polynomial f, but these conditions may not be satisfied if we choose a "truly" random polynomial. However, how can it be guaranteed that the pseudorandom measures of the sequence should be small for almost "random" polynomials? These semirandom polynomials will be constructed with as few modifications as necessary from a truly random polynomial.

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1 Introduction

Mauduit and Sárközy [11] proposed a new quantitative techniques to study pseudorandomness of binary sequences in 1997. They introduced the following new measures of pseudorandomness:

Definition 1 *For a binary sequence*

$$E_N = \{e_1, \dots, e_N\} \in \{-1, +1\}^N,$$

define the well-distribution measure of E_N *as*

$$W(E_N) = \max_{a,b,t} \left| \sum_{j=1}^{t} e_{a+jb} \right|,$$

where the maximum is taken over all a, b, t *such that* $a \in \mathbb{Z}$, $b, t \in \mathbb{N}$ *and* $1 \le a \le a+tb \le N$, *while the correlation measure of order* ℓ *of* E_N *is defined as*

$$C_\ell(E_N) = \max_{M,D} \left| \sum_{n=1}^{M} e_{n+d_1} \dots e_{n+d_\ell} \right|,$$

where the maximum is taken over all $D = (d_1, \dots, d_\ell)$ *and* M *such that* $0 \le d_1 < \dots < d_\ell < M + d_\ell \le N$.

It was an important topic in developing the theory of pseudorandomness what estimates may be given for W and C_ℓ for an average sequence. Cassaigne, Mauduit, and Sárközy [4] proved that for almost every binary sequence of length N,

$$\sqrt{N} \ll W(E_N) \ll \sqrt{N \log N}$$

and

$$\sqrt{N} \ll C_\ell(E_N) \ll \sqrt{\ell N \log N}.$$

Alon, Kohayakawa, Mauduit, Moreira and Rödl [2] sharpened the lower estimate with a factor $\sqrt{\log N}$ and the upper estimate with a constant factor,

giving the exact expected magnitude of these measures. Based on these results, it is safe to conclude that a sequence has very strong pseudorandom properties if

$$W(E_N),\ C_\ell(E_N) \ll \sqrt{N}\,(\log N)^c\,.$$

However, in practical applications, to have the estimates

$$W(E_N),\ C_\ell(E_N) \ll N^c$$

with a positive constant $c(<1)$ (as $N \to \infty$) is usually satisfactory. It should be noted that in practical applications a lower estimate is not required at all. As Alon, Kohayakawa, Mauduit, Moreira and Rödl [1] proved for even order correlation measure we always have

$$C_{2\ell}(E_N) \gg \sqrt{N}.$$

Although the correlation of odd order can be very small, even 1, it is clear from Gyarmati's [6] and later Anantharam's [3] and Gyarmati and Mauduit's [8] estimates that requiring a lower estimate is unnecessary.

The Legendre symbol sequence was the first to be studied using the measures defined above. Namely, let

$$E_{p-1} = \left\{ \left(\frac{1}{p}\right), \left(\frac{2}{p}\right), \left(\frac{3}{p}\right), \ldots, \left(\frac{p-1}{p}\right) \right\}.$$

Sárközy and Mauduit [11] proved that:

$$\begin{aligned} W(E_{p-1}) &\ll p^{1/2}\log p, \\ C_\ell(E_{p-1}) &\ll \ell p^{1/2}\log p. \end{aligned} \tag{1}$$

We can say that the Legendre sequence possesses strong pseudorandom measures because these estimates are substantially sharper than the trivial estimate.

This construction has one major drawback: it provides only one sequence for each prime. Hoffstein and Liemann [9] improved on this with a simple idea. Their construction was the following:

Construction 1 (Hoffstein, Liemann) *Let p be a prime, $f(x) \in \mathbb{F}_p[x]$ be a polynomial of degree k, which is not of the form $cg(x)^2$ with $c \in \mathbb{F}_p$, $g(x) \in \mathbb{F}_p[x]$. Define $E_p = (e_1, \ldots, e_p)$ by:*

$$e_n = \begin{cases} \left(\frac{f(n)}{p}\right) & \text{for } (f(n), p) = 1, \\ +1 & \text{for } p \mid f(n). \end{cases}$$

Hoffstein and Lieman, on the other hand, did not prove anything about the pseudorandomness of this sequence; they only claimed that it possesses strong pseudorandom properties. However, Goubin, Mauduit, and Sárközy [5] thoroughly studied Construction 1 and proved the following:

Theorem 1 (Goubin, Mauduit, Sárközy) *Let p be a prime, $f(x) \in \mathbb{F}_p[x]$ be a polynomial of degree k, which is not of the form $cg(x)^2$, where $c \in \mathbb{F}_p$, $g(x) \in \mathbb{F}_p[x]$. Define $E_p = (e_1, \ldots, e_p)$ by Construction 1. Then*

$$W(E_p) \ll kp^{1/2} \log p.$$

Assume that one of the following three conditions holds for the order ℓ of the correlation:
(i) $\ell = 2$;
(ii) $\ell < p$ and 2 is a primitive root modulo p;
(iii) $(4k)^\ell < p$.
Then, we have:

$$C_\ell(E_p) \ll k\ell p^{1/2} \log p.$$

Goubin, Mauduit, and Sárközy [5] also showed the existence of polynomials f such that the associated sequences have large correlation. Thus one of the conditions above, or a condition similar to those is really necessary to imply that the related sequence has strong pseudorandom properties.

As a result, the pseudorandom measures of the sequences given in Construction 1 are optimal, the elements of the sequence can be generated quickly, and the construction is natural. It is without a doubt one of the

most effective pseudorandom generators ever developed (further pseudorandom constructions, as well as their comparison, can be found, for example, in the survey paper [7]).

If the prime p and the coefficients of the polynomial f are given, the sequence in Construction 1 can be programmed quickly. When a sequence in Construction 1 is used as a secret key in cryptographic systems, the polynomial f must be chosen "almost" at random. This is significant because, for example, if the coefficients in the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

are consecutive integers, then the value of the other coefficients is immediately derived from the value a_n. Thus, simply looking at the cases $a_n = 1, 2, 3, ..., p-1$, the value of the sequence used as the secret key becomes decipherable. Similar problem can happen when f only has a few non-zero coefficients, in which case the secret key can be decrypted again using brute force.

Returning to the sequences in Construction 1, it is important that the polynomial f should be chosen at almost random. It is also important that k, the degree of the polynomial f should not be too small, since all sequences based on a polynomial of degree k or less can be programmed in a reasonable amount of time if the degree is small, and then our key sequence is no longer secret. As a result, in Construction 1, users need to choose the degree of the polynomial f for at least p^{ε} for some small positive constant ε. We believe that $\varepsilon = 0.1$ is ideal for applications, for example. However, if the degree is large, condition (iii) of Theorem 1 does not apply. Furthermore, if 2 is not a primitive root mod p, condition (ii) does not hold, and a high-order correlation measure can be very large. By Artin's conjecture 2 is a primitive root for infinitely many primes, but this is unproved yet.

We propose the following strategy: we select a random polynomial $f(x) \in \mathbb{F}_p[x]$. Then we look for a quadratic non-residue n for which $f(x)$ has no

irreducible factor of the form $(x+c)^2 - n$, with $c \in \mathbb{F}_p$. We also select a random $a \in \mathbb{F}_p$. Then, in place of f in Construction 1, we use the polynomial $g(x) = ((x+a)^2 - n)f(x)$.

Since f is a random polynomial, we can say that g is "semirandom". The only thing we know about it is that it has at least one quadratic irreducible factor. We will show that the pseudorandom measures of the sequences in Construction 1 based on the new polynomial g are optimally small. Then we create a fast (polynomial time) algorithm for calculating an appropriate n quadratic non-residue. Although our algorithm will be probabilistic rather than deterministic, it will not fail in practice. The probability of never finding a suitable quadratic non-residue n after running the Step 1-Step 6 of the algorithm 200 times is $< \frac{1}{2^{100}}$, which is extremely small.

In the following theorem, there is no need for any condition on the order of correlation (in contrast to Theorem 1, where there was a condition, see i), ii), or iii)), since the polynomial $g(x)$ has an irreducible quadratic factor that is not equivalent to any other factors.

Theorem 2 *Let p be a prime, $a \in \mathbb{F}_p$, n be a quadratic non-residue modulo p, and $f(x) \in \mathbb{F}_p[x]$ be a polynomial of degree k. If $f(x)$ has no irreducible factor of the form $(x+c)^2 - n$, with $c \in \mathbb{F}_p$, then define the polynomial $g(x)$ by:*

$$g(x) = ((x+a)^2 - n)f(x).$$

Furthermore, the sequence $E_p = \{e_1, e_2, e_3, \ldots, e_p\}$ is defined in the same way as in Construction 1, but with $g(x)$ within the Legendre symbol:

$$e_n = \begin{cases} \left(\frac{g(n)}{p}\right) & \text{for } (g(n), p) = 1, \\ +1 & \text{for } p \mid g(n). \end{cases}$$

Then:

$$W(E_n) \ll kp^{1/2} \log p,$$
$$C_\ell(E_n) \ll k\ell p^{1/2} \log p.$$

Proof of Theorem 2. Goubin, Mauduit, and Sárközy [5] proposed the following equivalence relation: The polynomials φ and $\psi \in \mathbb{F}_p[x]$ are equivalent if there exists $c \in \mathbb{F}_p$ for which

$$\varphi(x) = \psi(x + c).$$

Since n is a quadratic non-residue modulo p, the polynomial $g(x)$ in Theorem 2 has the irreducible factor $(x+a)^2 - n$. It is easy to see that there is no other irreducible factor that is equivalent to $(x + a)^2 - n$. Thus $g(x)$ is not of the form $c^* g^*(x)^2$ with $c^* \in \mathbb{F}_p$ and $g^*(x) \in \mathbb{F}_p[x]$. Furthermore, we know that for $1 \le d_1 < d_2 < \cdots < d_\ell \le p$, the polynomial $g(x + d_1)g(x + d_2) \cdots g(x + d_k)$ is not of the form $c^* g^*(x)^2$ with $c^* \in \mathbb{F}_p$ and $g^*(x) \in \mathbb{F}_p[x]$, since the factors equivalent to $(x + a)^2 - n$ in this product are:

$$(x + a + d_1)^2 - n, (x + a + d_2)^2 - n, \ldots, (x + a + d_\ell)^2 - n.$$

Each of the above irreducible factors appears exactly once in the decomposition of the polynomial $g(x + d_1)g(x + d_2) \cdots g(x + d_k)$ into irreducible factors.

We then use Weil's theorem [14] for the prime p and the Legendre symbol character:

Lemma 1 (Weil) *Suppose that $\mathbb{F}_q$ is a finite field, χ is a non-principal character of order d over it, $f \in \mathbb{F}_q[x]$ has s distinct roots in $\overline{\mathbb{F}}_q$, and it is not a constant multiple of the d-th power of a polynomial over $\mathbb{F}_q$. Then:*

$$\left| \sum_{n \in \mathbb{F}_q} \chi(f(n)) \right| \le (s - 1)q^{1/2}.$$

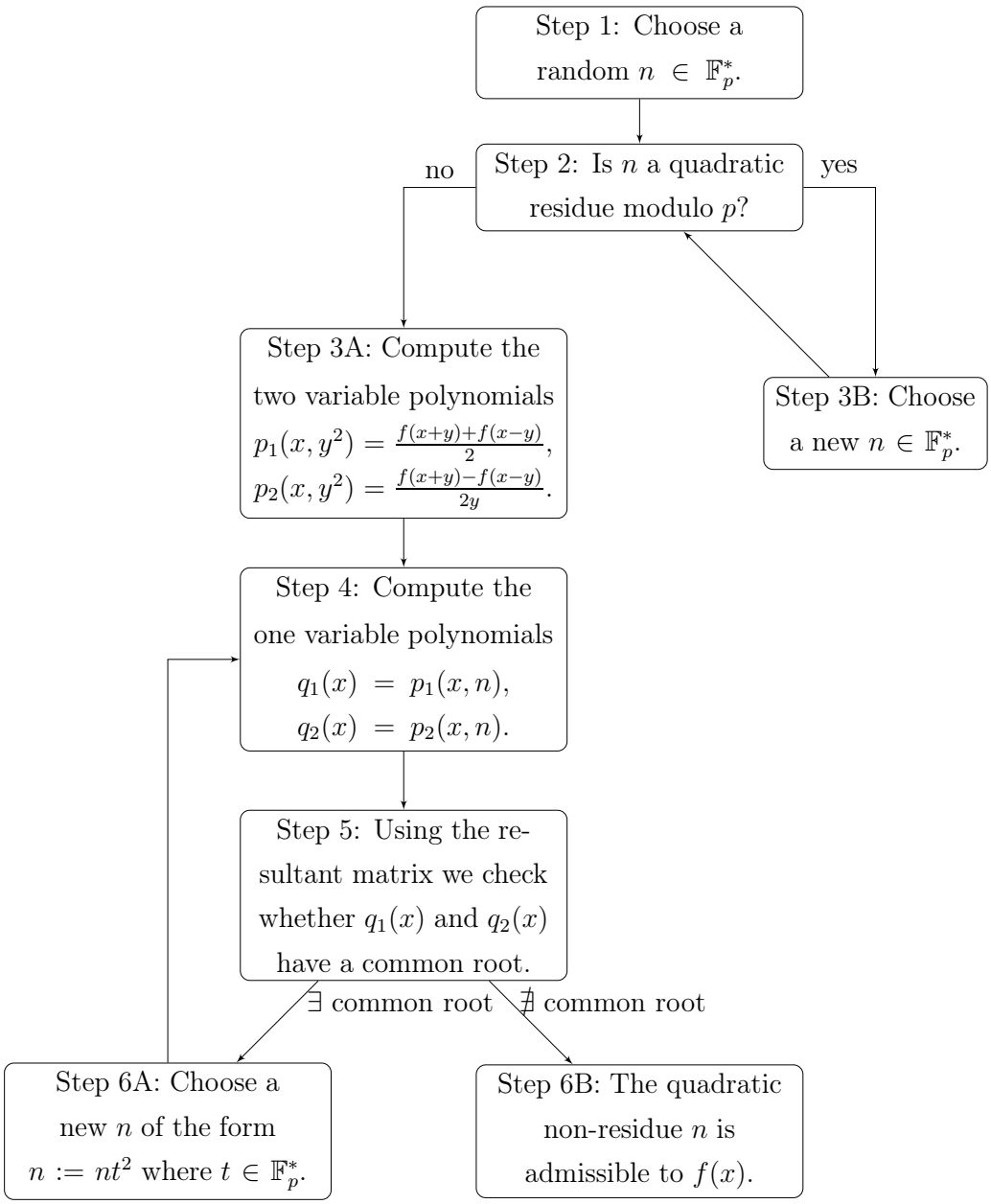
From here, the argument is the same as in the paper of Goubin, Mauduit, and Sárközy [5]; it is based on Weil's theorem above, and the estimates for $W(E_p)$ and $C_\ell(E_p)$ are obtained immediately. This completes the proof of Theorem 2.

2 Admissible quadratic non-residues and the algorithm

However, what conditions on the quadratic non-residue n are required to ensure that the polynomial $f(x)$ does not have an irreducible $(x+a)^2 - n$-shaped factor, where $a \in \mathbb{F}_p$? At first glance, this may appear to be a difficult question, but it is much easier to say that a polynomial does not have a specific type of root than it is to find one of the roots. We will now present a fast (polynomial-time) algorithm for finding an appropriate n quadratic non-residue. First we introduce a new definition.

Definition 2 *Let p be an odd prime, $f(x) \in \mathbb{F}_p[x]$ be a polynomial and n be a quadratic non-residue modulo p. If $f(x)$ has no irreducible factor of the form $(x+c)^2 - n$, with $c \in \mathbb{F}_p$, then the quadratic non-residue n is said to be admissible to $f(x)$.*

Now we will describe our algorithm for determining an n admissible to $f(x)$. The algorithm is first illustrated in a figure, followed by step-by-step instructions for each step of the algorithm.



This algorithm ends if it states that n is admissible to f, however there are many n's for which the algorithm cannot determine whether or not n is admissible, in which case a new n must be picked.

Theorem 3 *The probabilistic algorithm shown in the picture above is efficient in the sense that it generates an admissible n in polynomial time.*

(Moreover, the production of a single quadratic non-residue is the only probabilistic-nature phase in the process.)

The fact that it is never stated that a particular n is **not** admissible is immaterial because the primary goal of the technique is to generate n's admissible. Following this, we will study the algorithm in detail, including its speed, storage, and probabilistic nature.

Proof of Theorem 3. We would like to begin the algorithm by generating a quadratic non-residue n. This is accomplished by using random techniques. In Step 1, a random $n \in \mathbb{F}_p$ is chosen. Since the number of quadratic residues and quadratic non-residues are both $\frac{p-1}{2}$, it has a 50% probability of n being a quadratic residue and a 50% probability of n being a quadratic non-residue. If n happens to be a quadratic residue, we choose a new $n \in \mathbb{F}_p^*$. The process is repeated until a quadratic non-residue is found. The probability that we will always find a quadratic residue n in 100 trials is quite low: $\frac{1}{2^{100}}$. As a result, we can almost certainly find a quadratic non-residue in a very short time.

Next we try to find an n quadratic non-residue that is admissible to f. To do this, calculate the two variables polynomials p_1 and p_2 given in Step 3A. For p_1 and p_2 to be well defined, all the coefficients of the polynomials

$$\frac{f(x+y)+f(x-y)}{2}$$

and

$$\frac{f(x+y)-f(x-y)}{2y}$$

must be in $\mathbb{F}_p$ and the exponent of every power of y must be even in these polynomials. The first statement is obvious, while to prove the second, we write $f(x)$ in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

Then

$$\frac{f(x+y)+f(x-y)}{2} = \sum_{i=0}^{n} a_i \frac{(x+y)^i + (x-y)^i}{2}, \tag{2}$$

$$\frac{f(x+y)+f(x-y)}{2y} = \sum_{i=0}^{n} a_i \frac{(x+y)^i - (x-y)^i}{2y}, \tag{3}$$

$$\tag{4}$$

According to the binomial theorem, the exponent of y in both polynomials in (2) and (3) is even. That is, the polynomials p_1 and p_2 are well defined by the formulas given in Step 3A.

$$p_1(x, y^2) = \frac{f(x+y)+f(x-y)}{2},$$
$$p_2(x, y^2) = \frac{f(x+y)-f(x-y)}{2y}.$$

A simple calculation shows that

$$f(x+y) = p_1(x, y^2) + y p_2(x, y^2), \tag{5}$$

$$f(x-y) = p_1(x, y^2) - y p_2(x, y^2). \tag{6}$$

Assume that for a fixed n quadratic non-residue, $f(x)$ has an irreducible factor of the form $(x+c)^2 - n$ with $c \in \mathbb{F}_p$. Since n is a quadratic non-residue it is easy to see that $\mathbb{F}_{p^2}$ has an element θ for which

$$\theta^2 = n.$$

Then $\mathbb{F}_{p^2} = \mathbb{F}_p(\theta)$. By $\theta \notin \mathbb{F}_p$, we get that in case

$$u + v\theta = 0, \qquad u, v \in \mathbb{F}_p$$

we have

$$u = 0, v = 0$$

Since $f(x)$ has the irreducible factor $(x+c)^2 - n$, we know that $-c+\theta$ and $-c-\theta$ are roots of $f(x)$. Then writing $x = -c$ and $y = \theta$ in (5) and (6)

we get the result

$$0 = f(-c+\theta) = p_1(-c,\theta^2) + \theta p_2(-c,\theta^2),$$
$$0 = f(-c-\theta) = p_1(-c,\theta^2) - \theta p_2(-c,\theta^2).$$

Thus it follows from our previous remark with $u = p_1(-c,\theta^2) = p_1(-c,n)$ and $v = \pm p_2(-c,\theta^2) = \pm p_2(-c,n)$ that

$$p_1(-c,n) = 0 \qquad \text{and} \qquad p_2(-c,n) = 0,$$

so that the polynomials q_1 and q_2 given in Step 4 have a common root $-c$ in this case. If the resultant of the two polynomials q_1 and q_2 is not zero in $\mathbb{F}_p$, then these polynomials have no common root, and hence $f(x)$ cannot have an irreducible factor of the form $(x+c)^2 - n$, thus n is admissible to $f(x)$. If the resultant of the two polynomials q_1 and q_2 is zero in $\mathbb{F}_p$, the polynomials have a common root (but it is far from certain that it is in $\mathbb{F}_p$). Since we do not know if n is admissible in this case, we choose a new quadratic non-residue n. Since every quadratic non-residue has the form nt^2 where $t \in \mathbb{F}_p^*$, we do not need to use probabilistic methods to create this new quadratic non-residue; simply we define the new n by $n := nt^2$ with a $t \in \mathbb{F}_p^*$. We return to Step 4 with this new n.

What is the probability that we will get to Step 6B after Step 5 for a randomly chosen quadratic non-residue n, i.e. the polynomials q_1 and q_2 do not have a common root? We know that if there is a common root, which we denote by a, then:

$$\begin{aligned} p_1(a,n) &= 0 \\ p_2(a,n) &= 0 \\ \theta^2 &= n \\ f(a+\theta) = p_1(a,n) + \theta p_2(a,n) &= 0 \\ f(a-\theta) = p_1(a,n) - \theta p_2(a,n) &= 0. \end{aligned}$$

So that f has two roots, $\alpha = a + \theta$ and $\beta = a - \theta$ for which

$$\theta = \frac{\alpha - \beta}{2},$$
$$n = \left(\frac{\alpha - \beta}{2}\right)^2.$$

Then

$$n \in \mathcal{F} \stackrel{\text{def}}{=} \left\{ \left(\frac{\alpha - \beta}{2}\right)^2 : \ \alpha, \beta \text{ are different roots of } f \right\}.$$

Clearly,

$$|\mathcal{F}| \leq \binom{\deg f}{2} < \frac{k^2}{2}.$$

Thus if $n \notin \mathcal{F}$, then the polynomials q_1 and q_2 have no common root, implying that the quadratic non-residue n is certainly admissible to f.

Since $\mathcal{F}$ has less than $k^2/2$ elements, thus if we proceed Step 4-Step 6 for at least $k^2/2$ different quadratic non-residues n (which is possible for $p > k^2$), then we will certainly find one of them which is admissible to f. We may also study the probabilistic nature of these steps: by using the upper bound $k^2/2$ for the number of elements in $\mathcal{F}$, we can see that if $p > 2k^2$, the chance of getting from Step 5 to Step 6B for a randomly chosen quadratic non-residue n is more than $1/2$. Thus if we run this part of the algorithm 100 times, we will almost certainly find a quadratic non-residue that is admissible to f. The time required for the algorithm is $O(k^3(\log p)^2)$, the storage required is $O(k^2 \log p)$.

3 The program code of the algorithm

The algorithm from previous section was implemented in Matlab [10]. Our program is able to find a quadratic non-residue n which is admissible to $f(x)$. We must first enter a prime number (p), and the code will determine whether or not it is truly prime. Then we must specify which polynomial will be used during the algorithm. At first we need the degree of polynomial,

and we save it and denote it by (m). When defining the polynomial, the coefficients are listed in separately and stored in a vector for later use.

After carrying out the required calculations with the given polynomial $(h_1(x))$ we can compute the two variable polynomials (p_1, p_2). In both polynomials, the exponent of y is even. At this point, we ask the user to enter an integer n that is less than p but greater than 1 and we enter a 'while' loop to repeat when the condition is true. In this loop, we must determine whether n is a quadratic non-residue modulo p. We can use the Jacobi(n, p) function from another M-file [13] where it is implemented for this check.

In the case when n is admissible let us compute the one variable polynomials with $y^2 = n$ substitutions. Using the resulting matrix, we determine whether these two (one variable) polynomials have a common root. If yes, we must choose a new n; otherwise, the quadratic non-residue n is admissible to $f(x)$. The program also prints the runtime, which is affected by the choice of n, p and $f(x)$ polynomial. See the MatLab source code to find a quadratic non-residue n which is admissible to $f(x)$ in GitHub site [12].

4 Multiple grades of security

Assume we have a prime p and a polynomial f, and the sequence $E_p = \{e_1, e_2, \ldots, e_p\}$ is given by Construction 1, that is

$$e_n = \begin{cases} \left(\frac{f(n)}{p}\right) & \text{for } (f(n), p) = 1, \\ +1 & \text{for } p \mid f(n). \end{cases}$$

The correlation of this sequence can be large, and we improve on this by finding the quadratic non-residue n admissible to f using the algorithm described in Section 2. The polynomial $g(x)$ is then defined by the formula

$$g(x) = ((x + a)^2 - n)f(x),$$

where a is any element of $\mathbb{F}_p$. Furthermore, the following formula defines the sequence $F_p = (f_1, f_2, \ldots, f_p)$.

$$f_n = \begin{cases} \left(\frac{g(n)}{p}\right) & \text{for } (g(n), p) = 1, \\ +1 & \text{for } p \mid g(n). \end{cases} \tag{7}$$

The new sequence F_p has small pseudorandom measures due to Theorem 2; nonetheless, but is there a method to find a weak point in this new construction that may cause problems in applications? A skilled code breaker may be able to find the values of a and n in the definition of the polynomial $g(x)$ (for example, by going through all p^2 cases) and then observe that the sequence E_p is the same as

$$\left\{ \left(\frac{(1+a)^2 - n}{p}\right) f_1, \left(\frac{(2+a)^2 - n}{p}\right) f_2, \ldots, \left(\frac{(p+a)^2 - n}{p}\right) f_p. \right\}$$

Thus while the pseudorandom measures of the sequence F_p are optimal, a very simple operation can be used to return to the original E_p sequence, which may have a large correlation. This can be eliminated by producing more admissible quadratic non-residues $n_1, n_2, \ldots, n_r$ and defining the polynomial $g(x)$ by

$$g(x) = ((x+a_1)^2 - n_1)((x+a_2)^2 - n_2) \cdots ((x+a_r)^2 - n_r) f(x),$$

where $a_1, a_2, \ldots, a_t$ are arbitrary elements of $\mathbb{F}_p$. The sequence F_p is defined in the same way as before by (7). All pairs (a_i, n_i) as $i = 1, 2, \ldots, r$ can take on too many values, making the strategy of the code breaker described in this section ineffective. In practice, we believe that the choice $r = 20$ is already safe. (This claim is supported by the fact that when $r = 20$ is being used, the time required in brute force attacks changes to the 20th power of the original one.)

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